Interpolating Satellite derived Wind Field Data using Ordinary Kriging, with application to the Nadir Gap

Michael R. Inggs, *Member, IEEE*, and Richard T. Lord, *Student Member, IEEE*
Department of Electrical Engineering, University of Cape Town, South Africa

Abstract— Dual swath satellite wind sensors are unable to obtain measurements directly below the satellite, creating a "nadir gap" centered on the sub-satellite track. This is true of active sensors such as the scatterometer, as well as the microwave radiometers, newly used for this purpose. In addition, sensor coverage in equatorial regions is incomplete. Using processed ERS-1 wind field data, it is shown in this paper that Ordinary Kriging is an appropriate technique to interpolate this "nadir gap".

I. Introduction

This work was inspired by a desire to try and improve the spatial coverage of wind sensors in equatorial regions, where polar orbiting sensors in sun-synchronous orbits display significant gaps in the daily coverage. It then became apparent that the technique could also be used to fill the "nadir gap", defined below.

Due to the inability of satellite scatterometers to obtain wind field measurements for incidence angles below 20°, the NASA Scatterometer (NSCAT) wind field swaths have a gap centered on the sub-satellite track where no wind measurements are available. This gap (called the "nadir gap") arises, because for incidence angles smaller than 20°, σ^0 is only weakly sensitive to wind speed and virtually insensitive to wind direction [14].

Figure 1 illustrates the NSCAT antenna illumination pattern on the ocean surface. The swaths on either side of the sub-satellite track are 600 km wide, separated by a 329 km wide gap.

Recent work by Wentz [16] has shown that it is possible to use spaceborne radiometer data to measure the wind speed and velocity over the ocean surface. Satellite radiometers have the advantage over scatterometers of being less bulky and hence more reliable and cost effective. However, as for the active sensors, radiometers cannot measure wind vectors for incidence angles between zero and twenty degrees, resulting in a similar "nadir gap" centered on the sub-satellite track.

The following interpolation and smoothing techniques for spatial data were investigated and compared:

1. Trend Surface Analysis.
2. Splines.

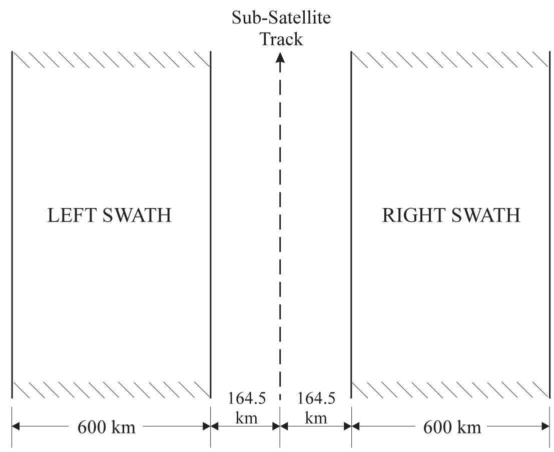

Figure 1: NSCAT antenna illumination pattern.

3. Kernel Smoothing.
4. Kriging.

Trend surface analysis [12] does not seem to be an appropriate method to interpolate wind field data, because of the difficulties in finding a simple functional form for the underlying trend of the data. Kernel smoothing methods [12] have the disadvantage of not being able to utilise the covariate information in the available data. It is also not obvious how the above methods can be used to interpolate vectors.

Both Kriging [12] and Spline based methods [12] seemed to be appropriate techniques to interpolate wind field vectors. Ordinary Kriging seemed most appropriate since it was specifically developed for interpolation in the case of random variables that exhibit spatial autocorrelation. Wind field data was expected to exhibit a high degree of autocorrelation. Furthermore, Ordinary Kriging has been successfully applied in a number of areas, such as soil mapping [4], mining, rainfall modelling and hydrology [12]. The use of splines to interpolate the nadir gap was only investigated theoretically. Spline based methods were never implemented to obtain interpolation results. Laslett [9] and Handcock *et al.* [3] discuss comparisons of the performance of Splines and Kriging in some applications.

The paper goes on to review Kriging briefly. To test the validity of the technique, artificial as well as ERS-1 wind field data was used. The nadir gap was simulated by removing a strip from the centre of the ERS-1 wind data. Good results were obtained and are reported here.

II. Kriging Wind Vectors

Kriging is a geostatistical interpolation technique which was first conceived of in the late fifties by D. G. Krige for use in the South African gold mining industry. It was formalized mathematically by G. Matheron [10] shortly thereafter. In this method the data is modelled as a stochastic process with a covariance function which is assumed stationary, that is, dependent only on distance and not on position. Kriging is often associated with the acronym B.L.U.E. for "best linear unbiased estimator". Kriging is "linear" because its estimates are weighted linear combinations of the sample values, it is "unbiased" since it tries to reduce the mean residual or error to zero, and it is "best" because it is the only estimation technique that aims at minimising the error variance [5]. Furthermore by explicitly modelling the covariance of the data points, this method is especially suited to data exhibiting spatial autocorrelation [12].

If we use the model

$$v_i = \tau_i + \eta_i \tag{1}$$

where τ represents large-scale underlying trend of the data, η represents the local spatially correlated component and the subscript i refers to the i th data point, where all the data points are numbered from 1 to n, then Kriging provides an estimator $\hat{v}_0$ of the form

$$\sum_{i=1}^{n} w_i v_i$$

where the weights w_i are chosen to minimize the estimation variance of the error, that is, to minimize

$$E\left[\left(\sum_{i=1}^{n} w_i v_i - v_0\right)^2\right]$$

where v_0 is the true value of the estimated variable at location $z_0 = (x_0, y_0)$.

In the case of *Simple Kriging*, the data are assumed to be detrended, so that the τ terms may be assumed to be zero. More generally, the trend term is either assumed to be a constant, as in *Ordinary Kriging*, or modelled as a polynomial in x and y, as in *Universal Kriging*. There are difficulties in finding a simple functional form for the underlying trend of wind data. (This was the key reason for rejecting trend surface analysis.) Hence, Universal Kriging does not seem to be an appropriate method to interpolate wind field data. Appendix A gives a more detailed description of the mathematical basis of Ordinary Kriging.

Wind field data consists of *vector* variables rather than scalars. According to Young [17], the Kriging technique can be extended to the spatial analysis of vector variables by defining the estimation variance and vector semi-variogram in terms of the magnitude of difference vectors. However the vectors have to be stationary, spatially correlated random variables.

The vector semi-variogram is defined as the expected value of the squared norm of the difference between two random vectors, which are, in turn, a vector distance $\mathbf{h}$ apart [17]:

$$\gamma(\mathbf{h}) = \frac{1}{2} E\left[\|\mathbf{v}_z - \mathbf{v}_{z+\mathbf{h}}\|^2\right] \tag{2}$$

where $\mathbf{v}_z$ is the regionalised vector variable at point $z = (x, y)$.

As for scalar variables, the vector semi-variogram can be estimated using

$$\gamma(\mathbf{h}) = \frac{1}{2n(\mathbf{h})} \sum \left\|\mathbf{v}_{z_i} - \mathbf{v}_{z_j}\right\|^2 \tag{3}$$

where the summation is over all $n(\mathbf{h})$ pairs which are a vector distance $\mathbf{h}$ apart.

This definition of the vector semi-variogram is consistent with the definition of the semi-variogram for scalar variables. The vector semi-variogram can apparently be used in classical geostatistical operations such as analysing anisotropic spatial variability and estimation variance [17]. For the remainder of this article, the word "semi-variogram" will refer to the *vector* semi-variogram, unless stated otherwise.

The estimation variance can be measured in various terms, such as the angular difference between the estimated vector $\hat{\mathbf{v}}_0$ and the actual vector $\mathbf{v}_0$, and some function of the angle, such as the cosine or tangent function. However Young [17] defines the estimation variance as the vector difference:

$$\mathrm{var}\left[\mathbf{v}_0 - \hat{\mathbf{v}}_0\right] = E\left[(\mathbf{v}_0 - \hat{\mathbf{v}}_0)^2\right] \tag{4}$$

This definition for the estimation variance is consistent with the vector semi-variogram, and can be minimized to yield the Kriging equations just as for the scalar case. The estimated vector can be expressed as a weighted linear combination of all the other vectors. Thus

$$\hat{\mathbf{v}}_0 = \sum_{i=1}^{n} w_i \mathbf{v}_i \tag{5}$$

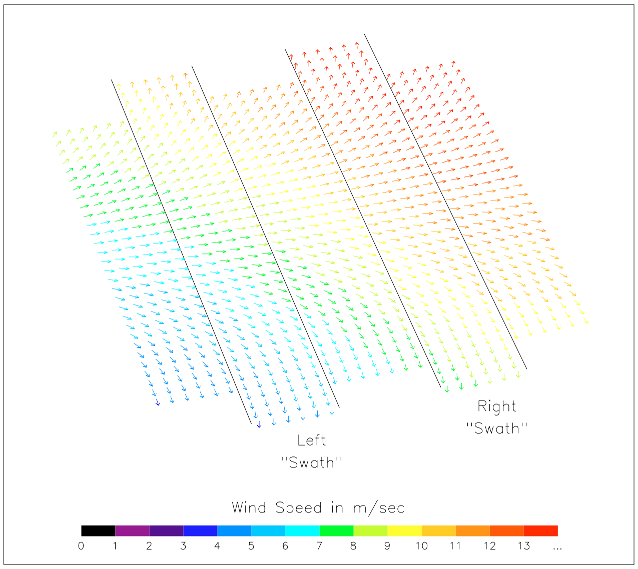

Figure 2: Interpolation results of synthetic data set.

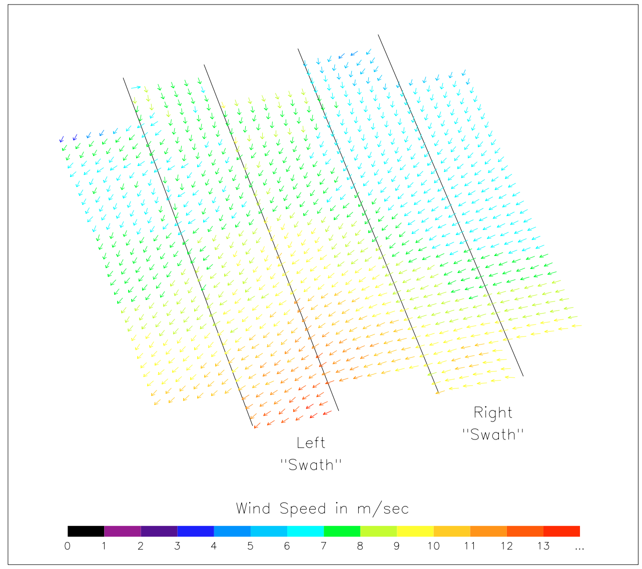

Figure 3: Interpolation results of real data set. © European Space Agency, ERS-1 data, 1994.

where the weights w_i are optimum in the sense of estimation variance, minimising the magnitude squared of the difference vector between the true and the estimated vector.

Before implementing this method on real ERS-1 wind field data, this method was tested using synthetic data sets. The "perfect" structure of synthetic data sets enables immediate evaluation of the obtained results. Figure 2 shows a synthetic data set that is 19 vectors wide and 38 vectors long. Both the wind speed and direction change from left to right and from the bottom to the top of the "swath". Seven vector columns were removed from the middle of the swath, creating a gap that was interpolated using Ordinary Kriging. From Figure 2 it can be seen that the results obtained look promising, with both the direction and speed of the estimated wind vectors following the basic trend of the original wind vectors. However the results obtained are not perfect and there are at least two reasons for this:

1. Ordinary Kriging assumes that the underlying trend of the data is constant, and this is not the case for this synthetic data set, which displays very obvious trends. The presence of an underlying trend was further confirmed by the parabolic shape of the semi-variograms.
2. The spherical model (see Section III-B), which was fitted to the calculated directional semi-variograms of this synthetic data set, did not fit very well, due to the parabolic shape of the semi-variograms.

These results do, however, indicate that Ordinary Kriging can be used to interpolate wind vectors. In the next section, processed ERS-1 wind field data is used to confirm these results.

III. Implementation on ERS-1 Data

A. Description of Implementation Procedure

The Kriging algorithm was implemented on seven data sets, which were all 19 wind vectors wide and 38 wind vectors long. These data sets were sections of longer ERS-1 wind field swaths. For each data set, 7 columns of vectors were removed from the middle of the swath. This gap was then interpolated, and the results were compared with the original strip.

When the swath was extrapolated, the semi-variogram was calculated and modelled using the entire data set, without the middle strip missing. However when the gap was interpolated, the semi-variogram was recalculated and modelled using all the data except the middle strip, which was assumed to be non-existent. This would be the situation when filling the "nadir gap".

Once the gap had been interpolated, the results were compared with the original values by calculating the rms value of the differences in the speed and angle values, and also the rms value of the magnitude of the difference vectors. The error in the speed component was also calculated as a percentage, by calculating the ratio between the speed rms value and the average speed in the *original* data strip.

For each interpolated vector the Kriging algorithm made use of 84 original vectors, with 42 vectors on either side of the gap. This number should be increased for better accuracy. However, the computation time will then also increase.

When the software was written to implement the Kriging algorithm, it was assumed that the wind vectors lie on a perfect grid. This is however not the case, as the horizontal and vertical distance between two successive wind vectors in degrees longitude and latitude changes slightly when moving from one end of the swath to the other. However, for swaths which are only 38 vectors long, this effect is negligible.

B. Interpreting Wind Field Semi-Variograms

All the results shown and discussed in this paper have been obtained by fitting a spherical semi-variogram model to the calculated semi-variograms. The spherical model is probably the most commonly used semi-variogram model, and is defined by the equation

$$\gamma(h) = \begin{cases} s\left[1.5\frac{h}{a} - 0.5\left(\frac{h}{a}\right)^3\right] & \text{if } 0 \le h \le a \\ s & \text{if } h > a \end{cases} \tag{6}$$

where s is the sill and a is the range. It has a linear behaviour at small separation distances near the origin, but flattens out at larger distances, and reaches the sill at a.

The wind field semi-variograms have been calculated in a direction parallel to the swath and in a direction perpendicular to the swath. Semi-variogram models often have different ranges and/or sills in different directions. For the case where only the range changes with direction, the anisotropy is known as *geometric anisotropy*, while in the case of only the sill changing with direction, the anisotropy is known as *zonal anisotropy* [5]. It has been found that the wind field data displays a mixture of geometric and zonal anisotropy. When modelling anisotropy, one usually starts by determining the anisotropy axes by experimentally determining the directions corresponding to the minimum and maximum range or sill. For this paper it has been assumed that the anisotropy axes correspond to the directions parallel to the swath and perpendicular to the swath. It is computationally much easier to find the semi-variogram for a swath in these two directions than in any other direction, and therefore the above assumption has been made. However it is possible to write a computer program to find the true anisotropy axes.

According to Isaaks and Srivastava [5], the isotropic model for two semi-variograms of the same type, but with range values of a_x and a_y, and sill values of w_1 and $(w_1 + w_2)$, can be given by

$$\gamma(\mathbf{h}) = w_1\,\gamma_1(h_1) + w_2\,\gamma_1(h_2) \tag{7}$$

where h_1 is defined as

$$h_1 = \sqrt{\left(\frac{h_x}{a_x}\right)^2 + \left(\frac{h_y}{a_y}\right)^2} \tag{8}$$

and h_2 is defined as

$$h_2 = \frac{h_y}{a_y} \tag{9}$$

where a_x and a_y are the ranges of the directional semi-variogram models along the axes of anisotropy and h_x and h_y are the components of $\mathbf{h}$ in the x and y directions of the anisotropy axes. It is important to note that the method described above is only appropriate for those situations where the directions of minimum and maximum continuity are perpendicular to one another. For this project it is assumed that this is the case for the wind field data. Isaaks and Srivastava [5] give further references that describe how to approach the modelling problem if the above method is not appropriate.

The range of influence of most wind field semi-variograms extends virtually over the whole width of the swath. The 7 interpolated columns of vectors fall well within the range of most wind field semi-variograms. It can therefore be concluded that the wind field data is highly correlated, indicating that the Kriging technique is appropriate to interpolate wind field data. However some wind field semi-variograms did not seem to reach a well-defined sill. This is indicative of a non-constant underlying trend, but Ordinary Kriging assumes that the underlying trend *is* constant. Universal Kriging would seem to be more appropriate, however the difficulty in finding a functional form

Table 1: RMS errors for real data using the spherical model

Swath	*Speed* [m/s]	*Angle* [Deg]	*Vector* [m/s]	*Avg Speed* [m/s]	*Speed* [%]
1	0.795	6.098	1.457	11.718	6.783
2	0.603	5.783	1.096	9.404	6.410
3	0.573	13.140	1.361	5.780	9.917
4	0.414	4.805	0.758	7.685	5.384
5	0.866	8.691	1.279	6.492	13.342
6	0.650	8.381	1.341	7.940	8.191
7	0.287	8.143	0.727	4.858	5.912
Average	0.598	7.863	1.146	7.697	7.991

that describes this trend makes Universal Kriging difficult to use. A technique that looks promising involves detrending the data before using Ordinary Kriging to estimate vectors, and then adding the trend back to the obtained result. Tevis *et al.* [15] describe a detrending technique called *median polish* which could possibly be applied to the wind field data before using Ordinary Kriging to estimate vectors. However Journel and Rossi [7] claim that it is only necessary to use a trend model in Kriging when one extrapolates data beyond the range of the semi-variogram. The large range values obtained for most wind field semi-variograms therefore suggest that the interpolation results obtained using Ordinary Kriging will not differ significantly from results obtained when using a trend model as in Universal Kriging.

A weighted non-linear least squares method was used to fit semi-variogram models to the calculated graphs. The points closer to the origin were given higher weights than points further away, because they are inherently more accurate, as they are calculated using more data pairs. A comparison between the results obtained using the spherical model, and the results obtained using the exponential model, showed that the spherical model yields better results on average. However the poorer results obtained from the exponential model were because the model-fitting algorithm sometimes produced extremely large range and sill values for the semi-variogram models. Semi-variograms that did not reach a well-defined sill produced extremely large range and sill values. When the semi-variogram parameters were estimated by "eye", with emphasis on fitting the bottom region of the graph reasonably well, both models yielded good results.

C. Interpolation Results

Table 1 summarises the rms errors obtained from interpolating the middle strip of the data sets. The average speed rms error obtained for the seven real data sets that were examined is of the order of 0.6 m/s or 8 %, which is acceptable, especially when one considers that the NSCAT system is required to measure wind speeds with an accuracy better than 2 m/s or 10 %, whichever is greater [14]. The average angle rms error is 7.863°, which is also acceptable. The NSCAT system is required to measure wind direction with an accuracy better than 20° [14]. The average angle rms error obtained here is therefore well within the range required for the NSCAT system.

D. Results obtained for Swaths of Data

When the same procedure was applied to whole swaths of data, the average speed rms error obtained was 10.069 %, which is slightly higher than the average of 7.991 % obtained for the real data sets. The average angle rms error was 12.178°, which is significantly higher than that obtained from the above data sets, which was 7.863°. This increase in the rms errors is probably due to underlying trend. The average speed and direction of wind vectors at the bottom of the swath is significantly different to that at the top of the swath. To minimize this effect, the Kriging algorithm should be applied to successive smaller sections of a swath, each section being about 38 vectors wide.

IV. Conclusion

The results obtained from interpolating synthetic and real data sets have shown that it is feasible to use Ordinary Kriging to interpolate wind field vectors.

Both the spherical and exponential semi-variogram models were used to obtain results. For both the synthetic and the real data sets, the spherical model yielded better results on average. However, it has been shown that fitting exponential models by "eye", with emphasis on fitting the bottom region of the graph very well, also yielded good results. Thus, the quality of the fitted semi-variogram model, especially near the lower region of the graph, plays a very important role in the accuracy of the results obtained.

Ordinary Kriging was also applied to wind field swaths to extrapolate vectors from either side of the swath, thereby further increasing ocean coverage. The results obtained looked very promising, however due to the difficulty in evaluating the accuracy of the extrapolated vectors, the results that were obtained have not been discussed in this paper.

The limits of the size of the gap that can be interpolated has not been rigorously investigated. The large range of influence of the semi-variograms does seem to suggest, however, that vectors estimated half a swath width away from the boundary of the swath are still reasonably accurate, given that enough samples have been used in the Kriging equations. It follows that the size of the gap that can be interpolated between two swaths should be less or equal to one swath width. A more conservative estimate would require the gap to be less or equal to two thirds of a swath width. The "nadir gap" of the NSCAT scatterometer, and possibly of future radiometer systems, which is less than a third of the swath widths on either side wide, can therefore be interpolated quite easily.

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REFERENCES

[1] I. Clark, *Practical Geostatistics.* London: Applied Science Publishers, 1979.

[2] N. Cressie, "Spatial Prediction and Ordinary Kriging," *Mathematical Geology*, vol. 20, no. 4, pp. 405–421, 1988.

[3] M. S. Handcock, K. Meier and D. Nychka, "Comments on 'Kriging and Splines: An Empirical Comparison of Their Predictive Performance in Some Applications'," *Journal of the American Statistical Association*, vol. 89, no. 426, pp. 401–403, June 1994.

[4] M. Inggs and R. Eatock, "The use of differential gps in field data extraction and spatially variable fertilizer application," *Proceedings of the 1994 International Geoscience & Remote Sensing Symposium (IGARSS '94)*, vol. 2, pp. 841–843, Pasadena, USA, August 94. IGARSS '94.

[5] E. H. Isaaks and R. M. Srivastava, *An Introduction to Applied Geostatistics.* New York: Oxford University Press, 1989.

[6] A. G. Journel, *Fundamentals of Geostatistics in Five Lessons,* Volume 8 of *Short Course in Geology.* American Geophysical Union, 1989.

[7] A. G. Journel and M. E. Rossi, "When do we need a trend model in kriging?," *Mathematical Geology*, vol. 21, pp. 715–739, 1989.

[8] K. A. Kelly and M. J. Caruso, "A Modified Objective Mapping Technique for Scatterometer Wind Data," *Journal of Geophysical Research*, vol. 95, no. C8, pp. 13483–13496, August 1990.

[9] G. M. Laslett, "Kriging and Splines: An Empirical Comparison of Their Predictive Performance in Some Applications," *Journal of the American Statistical Association*, vol. 89, no. 426, pp. 391–400, June 1994.

[10] G. Matheron, "Principles of Geostatistics," *Econ. Geology*, vol. 58, pp.1246–1266, 1963.

[11] P. C. McIntosh, "Oceanographic Data Interpolation: Objective Analysis and Splines," *Journal of Geophysical Research*, vol. 95, no. C8, pp. 13529–13541, August 1990.

[12] L. McNeill, "Topics in Interpolation and Smoothing of Spatial Data," PhD thesis, University of Cape Town, 1994.

[13] C. E. Mendoza, "Smoothing Unit Vector Fields," *Mathematical Geology*, vol. 18, no. 3, pp. 307–322, 1986.

[14] F. M. Naderi, M. H. Freilich and D. G. Long, "Spaceborne Radar Measurement of Wind Velocity Over the Ocean—An Overview of the NSCAT Scatterometer System," *Proceedings of the IEEE*, vol. 79, no. 6, pp. 850–866, June 1991.

[15] J. W. Tevis, A. D. Whittaker, and D. McCauley, "Efficient use of data in the Kriging of soil pH," *American Society of Agricultural Engineers*, pp. 1–17, 1991.

[16] F. J. Wentz, "Measurement of Oceanic wind vector using satellite microwave radiometers," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 30, no. 5, pp.960–972, September 1992.

[17] D. S. Young, "Random Vectors and Spatial Analysis by Geostatistics for Geotechnical Applications," *Mathematical Geology*, vol. 19, no. 6, pp. 467–479, 1987.

[18] D. L. Zimmerman and M. B. Zimmerman, "A Comparison of Spatial Semivariogram Estimators and Corresponding Ordinary Kriging Predictors," *Technometrics*, vol. 33, no. 1, pp. 77–91, February 1991.

Michael R. Inggs (M'82) received the B.Sc. (Hons.) degree in physics and applied mathematics from Rhodes University, South Africa, in 1973, followed by the Ph.D. degree from the Imperial College, London University, in 1979. He has worked in the areas of propagation in random media, microwave components and systems, and radar signal processing. He has been an Associate Professor in the Department of Electrical Engineering, University of Cape Town, South Africa, since 1988, where he leads a large postgraduate group studying imaging radar and radar signal processing.

Dr. Inggs is a registered professional engineer in South Africa.

Richard Thomas Lord (S'94) received the B.Sc. (Eng.) degree in electrical engineering from the University of Cape Town, South Africa, in 1994.

He is currently studying towards the Ph.D. degree in electrical engineering, working on radar remote sensing and image processing applications.

Mr. Lord is a student member of the SAIEE.

Appendix A: Ordinary Kriging

In Ordinary Kriging the estimated value is a weighted linear combination of the available data. Thus

$$\hat{v}_0 = \sum_{i=1}^{n} w_i v_i \tag{10}$$

where $\hat{v}_0$ is the estimated value at location z_0. The variables $w_1, \ldots, w_n$ are the weights and $v_1, \ldots, v_n$ are the original values. The subscript i refers to the i th data point, where all the data points are numbered from 1 to n. To ensure that the estimated value will be unbiased, the following constraint is introduced:

$$\sum_{i=1}^{n} w_i = 1 \tag{11}$$

The statistical approach to solve this problem is to model the available data and the unknown estimates as the outcome of a stationary random process. If the true value at location z_0 is v_0, then the expected error R will be:

$$R = E[\hat{v}_0 - v_0] \tag{12}$$

$$= E\left[\sum_{i=1}^{n} w_i v_i - v_0\right] \tag{13}$$

The estimation variance of the error is defined as:

$$\sigma_E^2 = E\left[\left(\sum_{i=1}^{n} w_i v_i - v_0\right)^2\right] \tag{14}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j c_{ij} - 2\sum_{i=1}^{n} w_i c_{i0} + c_{00} \tag{15}$$

where $c_{ij} = \text{cov}(v_i, v_j)$, the covariance of v_i and v_j.

Ordinary Kriging aims at minimising σ_E^2, the variance of the error. This is accomplished by setting the n partial first derivatives with respect to the w_i to zero. This produces a set of n equations in n unknowns. However, because of the constraint introduced in equation 11, the Lagrange multiplier technique has to be used to solve the system of n equations.

Rewriting equation 15 gives:

$$\sigma_E^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j c_{ij} - 2\sum_{i=1}^{n} w_i c_{i0} + c_{00} + 2\mu \underbrace{\left(\sum_{i=1}^{n} w_i - 1\right)}_{=0} \tag{16}$$

where μ is the Lagrangian multiplier. Taking the partial first derivative of σ_E^2 with respect to each of the w_i and μ, and equating the derivatives to zero gives the equation:

$$\underset{\mathbf{C}}{\begin{bmatrix} c_{11} & \cdots & c_{1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n1} & \cdots & c_{nn} & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix}} \cdot \underset{\mathbf{w}}{\begin{bmatrix} w_1 \\ \vdots \\ w_n \\ \mu \end{bmatrix}} = \underset{\mathbf{D}}{\begin{bmatrix} c_{10} \\ \vdots \\ c_{n0} \\ 1 \end{bmatrix}} \tag{17}$$

Solving for the weights gives:

$$\mathbf{w} = \mathbf{C}^{-1}\mathbf{D} \tag{18}$$

In practice the covariances are found indirectly via the semi-variogram. Although the kriging equations can be written in terms of the semi-variogram, it is computationally advantageous to use the covariance, since then the largest elements of the covariance matrix will be located on the diagonal. This leads to greater numerical stability for algorithms based on Gaussian elimination.

The semi-variogram function is defined as

$$\gamma(\mathbf{h}) = \frac{1}{2} E\left[(v_z - v_{z+\mathbf{h}})^2\right] \tag{19}$$

and can be found using the estimator

$$\gamma(\mathbf{h}) = \frac{1}{2n(\mathbf{h})} \sum (v_{z_i} - v_{z_j})^2 \tag{20}$$

where the summation is over all $n(\mathbf{h})$ pairs which are a vector distance $\mathbf{h}$ apart.

If there is greater spatial continuity in one direction than in another, the semi-variogram will have to be calculated for each direction. Often there is no directional effect and then one only needs to consider the distance $h = \|\mathbf{h}\|$.

Having found the semi-variogram, it is usual to fit one of several standard parametric models to the data [12]. The most commonly used models are the *nugget, linear, spherical, exponential* and *Gaussian* models.

The corresponding covariance which is used in the solution of the Kriging equations can be readily obtained from the following relationship:

$$C(h) = C(0) - \gamma(h) \tag{21}$$

where $C(0)$ can either be set equal to the sill $\gamma(\infty)$ of the semi-variogram model if it exists, or it can be set to any arbitrary large value [6]. The h in Equation 21 is the distance between the i th and j th point, where $C_{ij} = C(h)$.